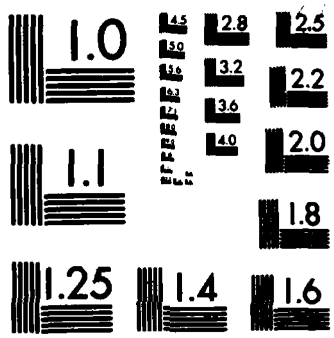
AD-A130 678 MIXED FINITE ELEMENT METHODS WITH APPLICATIONS TO FLOW AND OTHER PROBLEMS(U) TENNESSEE UNIV KNOXVILLE DEPT OF MATHEMATICS M D GUNZBURGER 31 MAY 83 AFOSR-TR-83-0624 1/1

UNCLASSIFIED AFOSR-80-0083 F/G 12/1 NL

END

FILMED

DTIC



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

# REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

(2)

| | |
|---|---|
| 1. REPORT NUMBER: AFOSR-TR- 83-0624 | 2. GOVT ACCESSION NO.: AD-A130678 |
| 3. RECIPIENT'S CATALOG NUMBER | |
| 4. TITLE (and Subtitle): Mixed Finite Element Methods with Applications to Flow and Other Problems | 5. TYPE OF REPORT & PERIOD COVERED: Final Report 1/1/80-3/31/83 |
| | 6. PERFORMING ORG. REPORT NUMBER |
| AUTHOR(s): Max D. Gunzburger | 8. CONTRACT OR GRANT NUMBER(s): AFOSR-80-0083 |
| PERFORMING ORGANIZATION NAME AND ADDRESS: Department of Mathematics, University of Tennessee, Knoxville, TN 37996 | 10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: 61102F 2304/A3 |
| CONTROLLING OFFICE NAME AND ADDRESS: AFOSR/NM, Bolling Air Force Base, DC 20332 | 12. REPORT DATE: 5/31/83 |
| | 13. NUMBER OF PAGES: 13 |
| MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) | 15. SECURITY CLASS. (of this report): Unclassified |
| | 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE |

DISTRIBUTION STATEMENT (of this Report)

**Approved for public release; distribution unlimited.**



17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Finite element methods, computational Fluid mechanics

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A report on the work carried out under the support of USAFOSR Grant No. AFOSR-80-0083 is given. The thrust of this work was the development of efficient and accurate finite element methods for flow problems. Specific applications include periodic acoustic problems, potential flow problems and incompressible viscous flows. However, the theoretical analyses carried out also have a direct bearing on the approximation of problems in other areas, e.g.,



83 07 26 . 101

FINAL REPORT FOR

GRANT AF-AFOSR-80-0083

(1/1/80 - 3/31/83)

MIXED FINITE ELEMENT METHODS WITH

APPLICATIONS TO FLOW AND

OTHER PROBLEMS

by

Max D. Gunzburger

Grant Institution: Department of Mathematics
University of Tennessee
Knoxville, TN 37996

Present Address: Department of Mathematics
Carnegie-Mellon University
Pittsburgh, PA 15213



May 31, 1983

TABLE OF CONTENTS

## I. ABSTRACT

We report on the work carried out under the support of USAFOSR Grant No. AFOSR-80-0083. The thrust of this work was the development of efficient and accurate finite element methods for flow problems. Specific applications include periodic acoustic problems, potential flow problems and incompressible viscous flows. However, the theoretical analyses carried out also have a direct bearing on the approximation of problems in other areas, e.g., electromagnetics and elasticity. For the particular fluids applications mentioned above, computer codes implementing the algorithms have also been developed.

# II. DESCRIPTION OF WORK ACCOMPLISHED

## A. Eigenvalue Problems

The first problem we consider is the energy stability of the incompressible Navier-Stokes equations. This problem has the following variational formulation: seek $(\underline{u},\varphi,\nu^*) \in V \times S \times \mathbb{R}$ such that

$$\begin{aligned} \int_\Omega (\underline{u} \cdot D \cdot \underline{w} - \varphi \operatorname{div} \underline{w}) &= -\nu^* \int_\Omega \nabla\underline{u} \cdot \nabla\underline{w} \\ \int_\Omega \psi \operatorname{div} \underline{u} &= 0 \end{aligned} \tag{1}$$

for all $(\underline{w},\psi) \in V \times S$. Here $V$ and $S$ are appropriately chosen Hilbert spaces and $D$ is the deformation tensor of a given flow. It is easily shown [1] that if the largest eigenvalue $\tilde{\nu}$ of the problem (1) is smaller than the kinematic viscosity $\nu$ of the given flow, then the given flow is stable in the energy sense. We note that (1) is a linear eigenvalue problem which determines stability regions for arbitrary perturbations, i.e., not infinitesimal. The problem (1) is, however, not easily solvable. We therefore wish to solve the problem approximately. To this end we choose finite dimensional subspaces $V^h \subset V$ and $S^h \subset S$ and require that (1) hold for all $(\underline{w},\psi) \in V^h \times S^h$. This leads to a linear generalized algebraic eigenvalue problem of the form

$$\begin{pmatrix} A & D^* \\ D & 0 \end{pmatrix} \begin{pmatrix} \vec{W} \\ \vec{\Phi} \end{pmatrix} = \nu_h \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \vec{W} \\ \vec{\Phi} \end{pmatrix}. \tag{2}$$

We choose the spaces $V^h$ and $S^h$ to be finite element spaces. Under certain conditions on these spaces, which are necessary for the stability of the approximate problem, we have been able to prove that the approximate eigenvalues $\nu_h$ converge to the exact eigenvalues $\nu^*$ at an optimal rate. In addition, two computer codes have been written which use different finite element spaces which satisfy the above mentioned conditions. The first uses a triangulation of the type illustrated in the figure on the left wherein the vector $\underline{u}$ is approximated by piecewise linear polynomials in each triangle and the

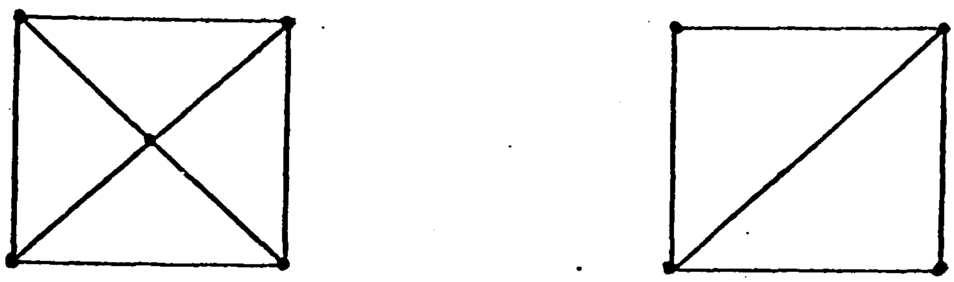


Figure 1

scalar $\varphi$ is approximated in a well defined subspaces of piecewise constant functions in each triangle. The second code uses the element on the right which again uses piecewise linear vectors on triangles, but now uses scalars which are constant throughout the quadrilateral. Our theory predicts that $\nu^h$ converges to $\nu^*$ quadratically in the grid spacing and, of course, this is verified by our computational results. The codes developed take advantage of the structure and sparsity of the algebraic problem (2).

Another physical problem considered is that of acoustic eigenvalue problems such as those described by the equation

$$\Delta\varphi = \lambda\varphi \quad \text{in} \quad \Omega \tag{3}$$

Again this eigenvalue problem can be given a variational characterization. We seek $(\underline{u},\varphi,\lambda) \in V \times S \times \mathbb{R}$ such that

$$\int_\Omega \psi \operatorname{div} \underline{u} = \lambda \int_\Omega \psi\varphi$$
$$\int_\Omega (\varphi \operatorname{div} \underline{v} + \underline{v} \cdot \underline{u}) = 0 \tag{4}$$

for all $(\underline{v},\psi) \in V \times S$, where again $V$ and $S$ are suitable Hilbert spaces. Approximations are defined as before. In this case we have proved optimal error estimates for the eigenvalues $\lambda$ and eigenfunction pairs $(\underline{u},\varphi)$ and have developed codes to compute these eigenvalues and eigenfunction, again using the two elements described in the figure above.

The variational problems (1) and (4) are special cases of the abstract mathematical problem: seek $(\underline{u},\varphi,\lambda) \in V_1 \times S_1 \times \mathbb{C}$ such that

$$\begin{aligned} a(\underline{u},\underline{v}) + b_1(\underline{v},\varphi) &= \lambda c(\underline{u},\underline{v}) \\ b_2(\underline{u},\psi) &= \lambda d(\psi,\lambda) \end{aligned} \tag{5}$$

for all $(\underline{v},\psi) \in V_2 \times S_2$. Here $a$, $b_1$, $b_2$, $c$ and $d$ are sesquilinear forms defined on the appropriate Hilbert spaces. We have been able to prove optimal error estimates for eigenvalue problems of the type (5) for a variety of combinations of forms $a$, $b_1$, etc., under reasonable hypothesis on these forms. For example, the cases $b_1 = b_2$ with $c$ or $d = 0$, $b_1 = b_2$ with both $c$ and $d \neq 0$, and $b_1 \neq b_2$, all with $a(\underline{u},\underline{v})$ coercive or weakly coercive have been successfully analyzed and illustrative computer programs have been implemented in each case. The various cases which have been considered describe a variety of physical examples including some generalized non-self-adjoint acoustic eigenvalue problems, eigenvalue problems emanating from transmission line theory, etc.

B. Inhomogeneous Navier-Stokes Equations

Finite element methods for stationary viscous incompressible flows were considered. Specifically, we consider the problem of finding a velocity field $\underline{u}$ and a pressure field $p$ which satisfy

$$\nabla \cdot \underline{u} = g \quad \text{in} \quad \Omega$$

$$\nu\Delta\underline{u} - \underline{u} \cdot \nabla\underline{u} + \nabla p = \underline{f} \quad \text{in} \quad \Omega \tag{6}$$

$$\underline{u} = \underline{q} \quad \text{on} \quad \Gamma$$

where $\Omega$ is a bounded region in $\mathbb{R}^2$ or $\mathbb{R}^3$ with boundary $\Gamma$, and where $g$, $\underline{f}$, and $\underline{q}$ are given functions. Previous work [2] on the topic considers the case $g = 0$ and $\underline{q} = 0$ only. The case $\underline{q} \neq 0$ is important since usually such flows are driven at the boundaries, e.g., by inflows. Our work on this topic included the following accomplishments:

a) Under natural hypothesis on $\underline{f}$, $g$, and $\underline{q}$, the existence and uniqueness of weak solutions of (6) was proven. Adding stability hypotheses on the finite element spaces, the existence and uniqueness of approximate finite element solutions was also displayed;

b) Optimal error estimates for the finite element discretization were derived;

c) The convergence behavior of iterative methods, i.e., the Newton, chord and simple iteration methods, for the solution of the discrete nonlinear algebraic equations resulting from the finite element discretization were analyzed. In particular, it was shown that the Newton method was quadratically convergent when one is close enough to the exact solution of the discrete equations, and it was also shown that a simple iteration scheme is globally convergent whenever the solution of (6) is unique; and

d) Computer programs were written implementing some particular choices of finite element spaces. These programs verified the theoretical results and also served to illustrate the implementation of the algorithms developed.

A crucial step in developing good algorithms for the approximation of the solution of (6) is choosing finite element spaces which satisfy the stability hypothesis alluded to in (a) above. We choose spaces $\underline{V}^h$ and $S^h$ in which to seek an approximation $\underline{u}^h$ and $p^h$, respectively, to the solution of (1). Then the stability hypothesis takes the form

$$\sup_{\underline{v}^h \in \underline{V}^h} \frac{\int_\Omega \psi^h \operatorname{div} \underline{v}^h d\Omega}{\|\underline{v}^h\|_1} \geq \gamma^h \|\psi^h\|_0 \quad \forall \psi^h \in S^h.$$

where $\|\cdot\|_1$ denotes the $H^1$-Sobolev norm. The crucial question is whether or not

$$\gamma^h \geq \gamma_0 > 0$$

uniformly in $h$, i.e., $\gamma_0$ is independent of $h$. For some obvious choices of $\underline{V}^h$ and $S^h$, $\gamma_0 = 0$ and these are discarded. For some other choices, such as bilinear velocities and constant pressures, $\gamma_0 \neq 0$ once the "checkerboard" pressure mode is removed. However, there is some evidence that $\gamma^h$ depends on $h$, i.e., $\gamma^h \to 0$. This results in possibly bad pressure approximations which must be filtered in order to obtain useful pressures. On the other hand, the velocity approximations are optimally accurate without any filtering. Computer programs implementing a variety of low order elements have been developed. These programs produce accurate velocity approximations.

### C. Mixed Finite Element Methods for Potential Flows

This work involves the approximation of problems of the type

$$\begin{aligned} \nabla \cdot \underline{u} &= F \quad \text{in} \quad \Omega \\ \underline{u} &= \nabla\phi \quad \text{in} \quad \Omega \\ \underline{u} \cdot \underline{n} &= f \quad \text{on} \quad \Gamma_N \\ \phi &= g \quad \text{on} \quad \Gamma_D \end{aligned} \tag{7}$$

where $\Gamma_N \cap \Gamma_D = \Gamma$, the boundary of $\Omega$, $f$, $g$ and $F$ are given functions, and $\underline{u}$ and $\phi$ are, for instance, an unknown velocity field and potential field. The simple problem (7) is equivalent to

$$\begin{aligned} \Delta\phi &= F \quad \text{in} \quad \Omega \\ \phi &= g \quad \text{on} \quad \Gamma_N \\ \frac{\partial\phi}{\partial n} &= f \quad \text{on} \quad \Gamma_0. \end{aligned} \tag{8}$$

However, we are interested in discretizing (7) directly, since in more general settings, problems of the type (7) may not always be recast into a form similar to (8). Previous work [3] on finite element methods for the approximation to the solution of (7) resulted in error estimates for the error $\underline{u} - \underline{u}^h$, $\underline{u}^h$ being the discrete solution, in the norm

$$\|\underline{v}\|_* = \|\underline{v}\|_0 + \|\mathrm{div}\ \underline{v}\|_0$$

where $\|\cdot\|_0$ denotes the $L^2$-norm. No error estimates were obtainable for $\|\underline{u}-\underline{u}^h\|_0$, which is physically of much greater interest than estimates for $\|\underline{u}-\underline{u}^h\|_*$. The thrust of our work was to examine conditions under which optimally accurate approximations in the norm $\|\cdot\|_0$ are obtainable for the problem (7). The highlights of this work are the following:

(a) Optimally accurate approximations are obtainable whenever the subspace $\underline{V}^h$ and $S^h$ in which we seek our approximate solution $\underline{u}^h$ and $\phi^h$, respectively, satisfy the inclusion property $S^h = \mathrm{div}(\underline{V}^h)$ and the decomposition property: every $\underline{u}^h \in \underline{V}^h$ may be written in the form

$$\underline{v}^h = \underline{w}^h + \underline{z}^h$$

where

$$\nabla \cdot \underline{z}^h = 0$$

and

$$\beta^h\|\underline{w}^h\|_0 \leq \|\mathrm{div}\ \underline{w}^h\|_{-1}$$

where

$$\|f\|_{-1} = \sup_{\substack{\psi \in H_0^1(\Omega) \\ \psi \neq 0}} \frac{\int_\Omega f\psi \, d\Omega}{\|\psi\|_1}.$$

Again, bounding $\beta^h$ by $\beta^h > \beta_0$ uniformly in $h$ is crucial to obtaining optimally accuate approximations;

(b) An example of a pair $\underline{V}^h$ and $S^h$ satisfying the above properties was given; and

(c) Computation using the spaces of (b) were performed, verifying the optimal theoretical results. Computations using finite element spaces which violate the conditions of (a) were also performed, and non-optimal or divergent approximations resulted.

D. <u>Problems with Inhomogeneous Essential Boundary Conditions</u>

Often, in application, we encounter problems with inhomogeneous boundary conditions. A simple example is provided by

$$\begin{aligned} -\Delta u &= f \quad \text{in} \quad \Omega \\ u &= g \quad \text{on} \quad \Gamma \end{aligned} \tag{9}$$

where $\Gamma$ is the boundary of the region $\Omega \subset \mathbb{R}^n$. Such problems present difficulties when finite element discretizations are considered, mainly because we cannot satisfy the boundary condition $u = g$ by functions in finite element spaces. We consider approximations of the solution of (9) wherein we first choose an approximation $g^h$, to $g$, which belongs to the restriction of the finite element space to the boundary. For example, $g^h$ could be a boundary interpolant of $g$ or an $L^2(\Gamma)$-projection of $g$. We then discretize (a), with $u = g$ replaced by $\tilde{u} = g^h$ by standard finite element techniques. We derive optimal $H^1$ and $L^2$ error estimate accounting for the differences between $g$ and $g^h$. Also, we show that such methods are easily implemented. The analyses and computer implementations were carried out for second order elliptic boundary valve problems, e.g., such as (9), and for the stationary Navier-Stokes equations.

E. Least Squares Finite Element Schemes

An important class of schemes which we have studied are least squares finite element schemes. Based on past analyses and experiences, we have developed least squares methods for a variety of flow applications. One interesting application is in transonic flows around harmonically oscillating wings. Such flows are governed by mixed type equations, i.e., of mixed hyperbolic/elliptic type. Assuming that the oscillations are small, one may derive a set of linear equations for the perturbation flow. These equations, which are analogous to the Helmholtz equations of acoustics, are again of the mixed type. We have developed and implemented a least squares finite element algorithm for the approximation of such flows. The most important features of this algorithm are its insensitivity to the type of the equation and the frequency of the oscillatory motion, and to the use of weighted inner products in order to accurately resolve singularities, e.g., at the leading edge.

## F. References

[1] Serrin, J., "Mathematical principles of classical fluid mechanics," Handbuch der Physick, 8 (1959), Springer.

[2] Girault, V. and P. Raviart, Finite Element Approximations of the Navier-Stokes Equation, (1979), Springer.

[3] Raviart, P. and J. Thomas, "A mixed finite element method for 2-nd order elliptic problems," Lecture Notes in Mathematics, No. 606, (1977), Springer.

## III. ACTIVITIES

PAPERS PREPARED UNDER GRANT SPONSORSHIP (Copies of these papers have been forwarded to AFOSR)

1. "Mixed finite element methods with applications to acoustic and flow problems," Proc. 5th AIAA Comp. Fluids Conf., AIAA CP814, pp. 265-271; [by G. Fix, M. Gunzburger, R. Nicolaides, J. Peterson].
2. "On mixed finite element methods for first order elliptic systems," Num. Math., 37, 1981, pp. 29-48; [by G. Fix, M. Gunzburger, R. Nicolaides].
3. "On conforming finite element methods for incompressible viscous flow problems," Comp. & Math. with Appls., 8, 1982, pp. 167-179; [by M. Gunzburger, R. Nicolaides, J. Peterson].
4. "On conforming finite element methods for the inhomogeneous stationary Navier-Stokes equations," to appear, Num. Math.; [by M. Gunzburger, J. Peterson].
5. "An application of mixed finite element methods to the stability of the incompressible Navier-Stokes equation, to appear, SIAM J. Scient. Stat. Comput.; [by J. Peterson].
6. "New results in the finite element solutions of steady viscous flows," The Mathematics of Finite Elements and Applications IV, Academic Press, 1982, pp. 463-470; [by M. Gunzburger and R. Nicolaides].
7. "On finite element approximations of problems having inhomogeneous essential boundary conditions," to appear, Comp. & Math. with Appls.; [by G. Fix, M. Gunzburger and J. Peterson].

8. "A least squares finite element scheme for transonic flow around harmonically oscillating wings," to appear, J. Comp. Phys.; [by C. Cox, G. Fix and M. Gunzburger].

## TALKS PRESENTED UNDER GRANT SPONSORSHIP

1. J. S. Peterson, "On mixed finite element methods for the stability of the Navier-Stokes equations," SIAM Fall Meeting, Houston, 1980.
2. M. D. Gunzburger and R. A. Nicolaides, "Mixed finite element methods for viscous flow problems," SIAM Fall Meeting, Houston, 1980.
3. M. D. Gunzburger, "Finite element approximations of the Navier-Stokes equations," AMS Meeting, Pittsburgh, 1981.
4. M. D. Gunzburger, "On mixed finite element methods for acoustic and flow problems," AIAA 5th CFD Conference, Palo Alto, 1981.
5. M. D. Gunzburger and R. A. Nicolaides, "Finite element methods for incompressible viscous flows," SIAM National meeting, Palo Alto, 1982.

## Ph.D. THESIS PARTIALLY SUPPORTED BY GRANT

Janet Peterson (Ph.D., 1980, Tennessee) Thesis title: "On mixed finite element methods for eigenvalue problems."

Jerome Eastham (Ph.D., 1981, Tennessee) Thesis title: "On the finite element method in anisotropic Sobolev spaces."

Georges Guirguis (Ph.D., Tennessee, expected in 1983) Thesis topic: "Finite element approximations to the Stokes equations in exterior domains."